RANKING OF FUZZY NUMBERS
AND
ITS APPLICATIONS

A MINOR PROJECT

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DECLARATION

I hereby affirm that the work **"Ranking of fuzzy numbers and its applications"** presented in this project is exclusively my own and there are no collaborators. All the ideas and references cited herein have been acknowledged.

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LIST OF PUBLICATIONS

1. Amit Kumar, S.S. Appadoo, **Parmpreet Kaur**, Mehar approach for solving dual-hesitant fuzzy transportation problem with restrictions, Sadhana – Academy Proceedings in Engineering Sciences, *DOI: 10.1007/s12046-020-01325-1* (Impact factor : 0.849).
2. **Parmpreet Kaur,** A new method for solving Pythagorean fuzzy transportation problems (Communicated).

CONTENTS

CHAPTER 1
Introduction

1.1 Origin of the problem

In general to find the optimal solution of a real-life problem, firstly, a mathematical model is obtained corresponding to the considered real-life problem. Then, an appropriate method is used to solve the obtained mathematical model. It is pertinent to mention that there exist several mathematical methods whose mathematical model represents a linear-programming problem.

Although, several methods have been proposed in the literature to solve linear programming problems. But, all these methods have been proposed by considering the assumption that each parameter of the obtained linear programming problem is precisely known. While, in real-life situations, some or all the parameters of the obtained linear programming problems may not be precisely known.

For example,

1. The fair of a cab between two fixed places depends upon the traffic jam or route followed by the cab or waiting time etc.
2. The availability of a product depends upon the various factors like weather condition, availability of transportation vehicle etc.
3. The demand of a product depends upon various factors like weather conditions, fluctuation in price etc.

To handle such situations, in the literature, the fuzzy set [20, 38] and its extensions [7] like dual-hesitant fuzzy set, Pythagorean fuzzy set etc. have been used to represent some or all the parameters of linear programming problems.

The linear programming problems in which some or all the parameters are represented as fuzzy numbers may be named as fuzzy linear programming problems. While, the linear programming problems in which some or all the parameters are represented by any extension of a fuzzy number may be named on the basis of the used extension of a fuzzy number e.g., a linear programing problem in which some or all the parameters are represented by dual-hesitant fuzzy set may be named as dual-hesitant fuzzy linear programming problem. While, a linear programming problem in which some or all the parameters are represented by a Pythagorean fuzzy number may be named as a Pythagorean fuzzy linear programming problem.

It is pertinent to mention that to find an optimal solution of a fuzzy linear programming problem, there is a need to rank (compare) the value of objective function corresponding to different basic feasible solutions i.e., there is a need to rank (compare) fuzzy numbers or extensions of a fuzzy number. Similarly, to find an optimal solution of a dual-hesitant fuzzy linear programming problem and to find an optimal solution of a Pythagorean fuzzy linear programming problem, there is a need to rank (compare) dual-hesitant fuzzy sets and Pythagorean fuzzy sets respectively.

Furthermore, it is necessary to mention that in general, to find the ranking of two or more fuzzy numbers/dual-hesitant fuzzy sets/ Pythagorean fuzzy set, firstly, an expression (named as Rank, Score, Accuracy etc.) the considered fuzzy numbers/dual-hesitant fuzzy sets/Pythagorean fuzzy numbers are firstly transformed into real numbers. Then, the transformed real numbers are used to finalize the ranking of considered fuzzy numbers/dual-hesitant fuzzy sets/Pythagorean fuzzy numbers.

Since, an expression to transform a fuzzy number/dual-hesitant fuzzy set/Pythagorean fuzzy number plays an important role in a method to solve a fuzzy/dual-hesitant, fuzzy/Pythagorean fuzzy linear programming problem. Therefore, it is necessary to check its validity as if the expression, used to transform a fuzzy number/dual-hesitant fuzzy set/Pythagorean fuzzy number into a real-number, will not be valid. Then, the optimal solution and the optimal value of the considered fuzzy/dual-hesitant fuzzy/ Pythagorean fuzzy linear programming problem will also not be valid.

After a deep study, it is observed that

(i) The expression (named as Score function), used by Maity et al. [29] to transform a dual-hesitant fuzzy set into a real-number, is not valid and hence, Maity et al.'s approach [29] to solve dual-hesitant fuzzy transportation problems is not valid.

(i) The expressions (named as Score function and Accuracy function), used by Kumar et al. [22] to transform a Pythagorean fuzzy number into a real-number, is not valid and hence, Kumar et al.'s approach [22] to solve Pythagorean fuzzy transportation problems is not valid.

The aim of this research is to point out the invalidity of the

(i) The expression (named as Score function), used by Maity et al. [29] to transform a dual-hesitant into a real-number. Also, to propose a valid expression (named as Mehar expression) to transform a dual-hesitant fuzzy set into a real-number.

Furthermore, to propose a valid approach (named as Mehar approach) to solve dual-hesitant fuzzy transportation problems.

(ii) The expressions (named as Score function and Accuracy function), used by Kumar et al. [22] to transform a Pythagorean fuzzy number into a real-number, are not valid. Also, to propose valid expressions to transform a Pythagorean fuzzy number into a real-number. Furthermore, to propose a valid approach to solve Pythagorean fuzzy transportation problems.

1.2 Literature Survey

Now a days, in competitive market, minimizing transportation cost becomes an important challenge. The basic transportation problem was developed by Hitchcock [17]. In real-life problems, it may be easily observed that the price of the same product varies at different places. This variation may occur due to several factors. Transportation cost is one of the common factors for this variation. The price of a product is directly proportional to the transportation cost i.e., price of product will increase/decrease with the increase/decrease in the transportation cost. Due to the same reason, it is necessary to determine the optimal way of supplying the product from various sources to various destinations. In general, the classical methods [1, 4, 5, 16, 37] (North west corner method, Least cost method, Vogel's approximation methods etc.) are used to find one of the possible way to transport the product and then the classical modified distribution method is applied to find the optimal way with the help of the obtained possible way to transport the product.

It is important to mention that all the above mentioned classical methods can be used only if the precise value of all the transportation parameters (Availability of the product at each source, demand of the product at each destination and the cost for supplying the one unit quantity of the product from each source to each destination) is known. However, in real-life situations these parameters are not precisely known.

Due to these facts, in the literature, fuzzy set [38] and its various extensions [7] have been used to represent various transportation parameters. Also, various methods have been proposed in literature to solve transportation problems under fuzzy environment and its various extensions.

In this section, a brief review of some methods for solving fuzzy transportation problems and its various extensions has been discussed.

Kaur and Kumar [18] proposed the fuzzy north-west corner method, fuzzy least cost method and fuzzy Vogel's approximation method to find the initial fuzzy basic feasible solution by using ranking function.

Kaur and Kumar [19] proposed the fuzzy north-west corner method, fuzzy least cost method and fuzzy Vogel's approximation method to find the initial fuzzy basic feasible solution as well as fuzzy modified distribution method to solve such transportation problems in which all the transportation parameters are represented by generalized trapezoidal fuzzy numbers.

Kumar and Kaur [24] pointed out the drawbacks of existing methods for solving transportation problems under fuzzy environment. Also, to resolve the drawbacks, Kumar and Kaur [24] proposed a method to solve such unbalanced transportation problems in which all the parameters are represented as trapezoidal fuzzy numbers. This method is based upon a fuzzy linear programming method which is obtained by replacing each parameter (represented by a non-negative real-number) of the transportation problem with a trapezoidal fuzzy number. In this method, firstly, the obtained fuzzy linear programming problem is transformed into its equivalent crisp linear programming problem. Then, the optimal solution of the transformed crisp linear programming problem is used to obtain the fuzzy optimal solution of the transportation problem under fuzzy environment.

Gupta and Kumar [14] extended Kumar and Kaur's method [24] to solve such multi-objective transportation problems in which all the parameters are represented by trapezoidal fuzzy numbers.

Ebrahimnejad [9] pointed out to the procedure of Kaur and Kumar's method [19], there is need to use arithmetic operations of fuzzy numbers and hence, much computational efforts are required to apply Kaur and Kumar's method [19]. To reduce the computational efforts, Ebrahimnejad [9] proposed a method to solve the same type of transportation problems. In this method, firstly, the considered generalized fuzzy transportation problem is transformed into its equivalent crisp transportation problem. Then, the optimal solution of the transformed crisp transportation problem is used to determine the optimal solution and the optimal generalized fuzzy transportation cost of the considered generalized fuzzy transportation problem.

Rani et al. [30] proposed a method to reduce the computational efforts of Kumar and Kaur's method [24]. In this method, firstly, the fuzzy linear programming problem of a transportation problem under fuzzy environment is transformed into its equivalent four crisp linear programming problems. Then, the optimal solutions of these crisp linear programming problems are used to find the fuzzy optimal solution of the transportation problem under fuzzy environment.

Singh and Yadav [33] proposed the intuitionistic fuzzy north-west corner method, intuitionistic fuzzy least cost method and intuitionistic fuzzy Vogel's approximation method to find the initial intuitionistic fuzzy basic feasible solution as well as intuitionistic fuzzy modified distribution method to find the optimal solution of such transportation problems in which the cost for transporting one unit quantity of the product from each source to each destination is represented by a triangular intuitionistic fuzzy number. While, all the remaining parameters are represented as non-negative real-numbers.

Singh and Yadav [34] proposed the intuitionistic fuzzy north-west corner method, intuitionistic fuzzy least cost method and intuitionistic fuzzy Vogel's approximation method to find the initial intuitionistic fuzzy basic feasible solution as well as intuitionistic fuzzy modified distribution method to find the intuitionistic fuzzy optimal solution of such transportation problems in which the availability of the product at each source and the demand at each destination is represented by a intuitionistic type 2 fuzzy number. While, Singh and Yadav [35] proposed a method for transportation problem in which each parameter is represented by intuitionistic fuzzy number.

Kumar and Hussain [23] proposed a method for solving such balanced transportation problems in which each transportation parameter is represented as a triangular intuitionistic fuzzy number. This method is based upon an intuitionistic fuzzy linear programming problem which is obtained by replacing each parameter (represented by a non-negative real-number) of the transportation problem with a triangular intuitionistic fuzzy number. In this method, firstly, the obtained intuitionistic fuzzy linear programming problem is transformed into its equivalent crisp linear programming problem. Then, the optimal solution of the transformed crisp linear programming problem is used to obtain the intuitionistic fuzzy optimal solution of the transportation problem under intuitionistic fuzzy environment.

Ebrahimnejad [10] pointed out that more than one fuzzy optimal transportation cost is obtained on applying Kumar and Kaur's method [24], which is mathematically incorrect. Ebrahimnejad [10] also pointed out that this drawback is occurring due to using the inappropriate function for comparing trapezoidal fuzzy numbers. To resolve the drawback, Ebrahimnejad [10] proposed a method, based upon a different function for comparing trapezoidal fuzzy numbers, to solve such balanced transportation problems in which each parameter is represented by a trapezoidal fuzzy number.

Ebrahimnejad [11] proposed a method to transform such an unbalanced transportation problems into a balanced transportation problem in which each transportation parameter is represented by a generalized interval-valued trapezoidal fuzzy number. Ebrahimnejad [11] also proposed a method to solve this type of transportation problems. This method is based upon a generalized interval-valued fuzzy linear programming problem which is obtained by replacing each parameter (represented by a non-negative real-number) of the transportation problem with a generalized interval-valued trapezoidal fuzzy number. In this method, firstly, the obtained generalized interval-valued fuzzy linear programming problem is transformed into its equivalent crisp linear programming problem. Then, the optimal solution of the transformed crisp linear programming problem is used to obtain the generalized interval-valued fuzzy optimal solution of the transportation problem under generalized interval-valued fuzzy environment.

Ebrahimnejad and Verdegay [12] proposed a method for solving such balanced transportation problems in which each transportation parameter is represented as a trapezoidal intuitionistic fuzzy number. This method is based upon an intuitionistic fuzzy linear programming problem which is obtained by replacing each parameter (represented by a non-negative real-number) of the transportation problem with a trapezoidal intuitionistic fuzzy number. In this method, firstly, the obtained intuitionistic fuzzy linear programming problem is transformed into its equivalent crisp linear programming problem. Then, the optimal solution of the transformed crisp linear programming problem is used to obtain the intuitionistic fuzzy optimal solution of the transportation problem under intuitionistic fuzzy environment.

Various attempts have been made by researchers to solve transportation problems in the fuzzy environment. In 1984, Chanas et al. [8] suggested fuzzy transportation problems. Many authors did lot of work in transportation problems by using various fuzzy environments such as integer fuzzy [36], multi-objective [14, 26], type-2 fuzzy [15, 25, 28, 31], interval and integer valued fractional fuzzy [2, 3, 6], intuitionistic fuzzy set [21]. Moreover, after reviewing literature it has been noticed that numerous methods have been proposed to solve this transportation problem by using extension principle [27], ranking function [18], modified Vogel's approximation method [32], Simplex type algorithm [13] and so on.

Maity et al. [29] pointed out that a supplier may have different types of vehicles to transport the product from each source to each destination. But, in general, it is assumed that the supplier will use that vehicle to transport the product corresponding to which the transportation cost will be minimum. However, this assumption is not realistic as in real-life situations, it is not always possible to transport the product with a vehicle having minimum transportation cost. Maity et al. [29] also pointed out that if a vehicle having minimum transportation cost is used to supply the product. Then, the supplier will be fully satisfied. However, if a vehicle having minimum transportation cost is not used to transport the product. Then, the supplier will be partially satisfied and partially unsatisfied and hence, a degree of satisfaction and degree of dissatisfaction may be associated with the transportation cost. The degree of satisfaction will decrease with increase in the transportation cost. While, the degree of dissatisfaction will increase with the increase in the transportation cost.

To handle such real-life transportation problems, Maity et al. [29] proposed the concept of dual-hesitant fuzzy transportation problems as well as a method to solve the dual-hesitant fuzzy transportation problems. In dual-hesitant fuzzy transportation problems, a degree of satisfaction and a degree of dissatisfaction is assigned with the transportation cost of each available vehicle e.g., let three vehicles be available to transport the product from the i^{th} source S_i to the j^{th} destination D_j and let the transportation cost corresponding to these vehicles be 30, 40 and 50. Furthermore, let the degree of satisfaction and the degree of dissatisfaction of the supplier corresponding to first, second and third vehicle be 0.5, 0.3, 0.2 and 0.4, 0.5, 0.7 respectively. Then, the transportation cost from the i^{th} source S_i to the j^{th} destination D_j may represented by the dual-hesitant fuzzy set $\{\{0.5, 0.3, 0.2\}, \{0.4, 0.5, 0.7\}\}(30, 40, 50\ )$.

Kumar et al. [22] proposed a method to solve such transportation problems in which the cost for supplying one unit quantity of the product from each source to each destination is represented by a Pythagorean fuzzy number. While, the remaining parameters are represented by non-negative real numbers. This method is based upon a Pythagorean fuzzy transportation table which is obtained by replacing the cost for supplying one unit quantity of the product (represented by a non-negative real-number) from each source to each destination of the transportation table with a Pythagorean fuzzy number. In this method, firstly, the obtained Pythagorean fuzzy transportation problem is transformed into a crisp transportation problem. Then, the classical methods (North west corner method, Least cost method, Vogel's approximation method etc.) are used to obtain one of the possible solution of these transportation problems and hence, modified distribution method is applied to find the optimal solution of the transformed crisp transportation problem. Finally, the optimal solution of the transformed crisp transportation problem is used to find the optimal solution of the transportation problem under Pythagorean fuzzy environment.

1.3 Chapter-wise summary of this research work

The chapter-wise summary of this research work is as follow:

Chapter 2

Mehar approach for solving dual-hesitant fuzzy transportation problem with restrictions

It is compulsory to mention that as there does not exist any other approach except Maity et al.'s approach [29] to solve the dual-hesitant fuzzy transportation problems. Therefore, in future, other researchers may use Maity et al.'s approach [29] to find the optimal solution of real-life dual-hesitant fuzzy transportation problems. However, after a deep study, it is observed that Maity et al.'s approach [29] is inappropriate. To validate this claim, in this chapter, two dual-hesitant fuzzy transportation problems are solved by Maity et al.'s approach [29] and shown that the obtained solutions are not appropriate. Also, it is pointed out that the inappropriateness in the obtained solutions is occurring due to using the inappropriate expression to transform a dual-hesitant fuzzy set into a real-number. Furthermore, to resolve the inappropriateness of Maityet al.'s approach [29], a new expression (named as Mehar score function) and an appropriate approach (named as Mehar approach), based upon the proposed Mehar score function, is proposed to find the optimal solution of dual-hesitant fuzzy transportation problems.

Chapter 3

A new method for solving Pythagorean fuzzy transportation problems

It is pertinent to mention that as there does not exist any other approach except Kumar et al.'s approach [22] to solve the Pythagorean fuzzy transportation problems. Therefore, in future, other researchers may use Kumar et al.'s approach [22] to find the optimal solution of real-life Pythagorean fuzzy transportation problems. However, after studying, it is observed that Kumar et al.'s approach [22] is inappropriate. To validate this claim, in this chapter, Pythagorean fuzzy transportation problem is solved by Kumar et al.'s approach [22] and shown that the obtained solutions are not appropriate. Also, it is pointed out that the inappropriateness in the obtained solutions is occurring because of using the inappropriate expression to transform a Pythagorean fuzzy number into a real-number. Furthermore, to resolve the inappropriateness of Kumar et al.'s approach [22], new expressions to transform a Pythagorean fuzzy number into a real-number and an appropriate approach to solve Pythagorean fuzzy transportation problems, are proposed.

1.4 Preliminaries

In this section, some basic definitions are presented.

Definition 1.4.1 [38] A set $\tilde{A} = \{\langle x, \mu_{\tilde{A}}(x)\rangle \mid x \in X,\ 0 \le \mu_{\tilde{A}}(x) \le 1\}$, defined on the universal set X, is said to be a Fuzzy Set, where $\mu_{\tilde{A}}(x)$ represents the degree of membership of the element x in $\tilde{A}$.

Definition 1.4.2 [29] Let X be an initial universe of objects. A set $\tilde{A}$ on X defined as $\tilde{A} = \{\langle x,\ \mu_{\tilde{A}}(x^{(s)})\rangle | x \in X\}$ is called a Hesitant fuzzy set, where $\mu_{\tilde{A}}(x^{(s)})$ is a mapping defined by

$$\mu_{\tilde{A}}(x^{(s)}): X \to [0,1]$$

here, $\mu_{\tilde{A}}(x^{(s)})$ is a set of some different values in [0,1] and $\boldsymbol{s}$ represents the number of possible membership degrees of the element $x \in X$ to $\tilde{A}$.

Definition 1.4.3 [29] A set $\tilde{A}$ on X defined as $\tilde{A} = \{\langle x,\ \mu_{\tilde{A}}(x^{(s)}), \nu_{\tilde{A}}(x^{(t)})\rangle | x \in X\}$ is called a Dual Hesitant fuzzy set,

where, $\mu_{\tilde{A}}(x^{(s)}), \nu_{\tilde{A}}(x^{(t)})$ is a mapping defined by

$$\mu_{\tilde{A}}(x^{(s)}), \nu_{\tilde{A}}(x^{(t)}): X \to [0,1],$$

here $\mu_{\tilde{A}}(x^{(s)}), \nu_{\tilde{A}}(x^{(t)})$ is a set of some different values in [0,1], $\boldsymbol{s}$ represent the number of possible membership degrees and $\boldsymbol{t}$ represent the number of possible non membership degrees of the element $x \in X$ to $\tilde{A}$.

Definition 1.4.4 [22] Let X is a fixed set, a Pythagorean fuzzy set (PFS) is an object having the form $P = \{\langle x, (\theta_P(x), \delta_P(x))\rangle | x \in X\}$

Where the function $\theta_P(x): X \to [0,1]$ and $\delta_P(x): X \to [0,1]$ are the degree of membership and non-membership of the element $x \in X$ to P, respectively. Also for every $x \in X$, it holds that $(\theta_P(x))^2 + (\delta_P(x))^2 \leq 1$

Chapter 2

Mehar approach for solving dual-hesitant fuzzy transportation problem with restrictions

Recently, a new type of transportation problem (named as dual-hesitant fuzzy transportation problem) as well as an approach to find the optimal solution of dual hesitant fuzzy transportation problems has been proposed in the literature. In this Chapter, some dual-hesitant fuzzy transportation problems are considered to show that the existing approach is inappropriate as (i) The existing approach fails to find the optimal solution of dual-hesitant fuzzy transportation problems (ii) On applying the existing approach different optimal transportation costs are obtained corresponding to alternative optimal solutions. Also, to resolve the inappropriateness of the existing approach, a new expression (named as Mehar score function) is proposed to transform a dual-hesitant fuzzy set into a real number. Furthermore, a new approach (named as Mehar approach), based upon the proposed Mehar score function, is proposed to find the optimal solution of dual-hesitant fuzzy transportation problems.

2.1 Maity et al.'s approach to find the optimal solution of dual-hesitant fuzzy transportation problems

Maity et al. [29] proposed the following approach to find the optimal solution of the dual-hesitant fuzzy transportation problem (represented by Table 2.1).

Table 2.1. Dual-hesitant fuzzy transportation problem

	D_1	D_2	$\cdots$	D_n	Availability a_i
S_1	$\tilde{c}_{11}$	$\tilde{c}_{12}$	$\cdots$	$\tilde{c}_{1n}$	a_1
S_2	$\tilde{c}_{21}$	$\tilde{c}_{22}$	$\cdots$	$\tilde{c}_{2n}$	a_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
S_m	$\tilde{c}_{m1}$	$\tilde{c}_{m2}$	$\cdots$	$\tilde{c}_{mn}$	a_m
Demand b_j	b_1	b_2	$\cdots$	b_n	

where,

(i) The dual-hesitant fuzzy set $\tilde{c}_{ij} = \left\{\{\gamma_{ij1}, \gamma_{ij2}, \dots, \gamma_{ijk}, \dots, \gamma_{ijp}\}, \{\eta_{ij1}, \eta_{ij2}, \dots, \eta_{ijk}, \dots, \eta_{ijp}\}\right\} \left(\{c_{ij1}, c_{ij2}, \dots, c_{ijk}, \dots, c_{ijp}\}\right)$ represents the cost for supplying the unit quantity of the product from the i^{th} source S_i to the j^{th}destination D_j.

(ii) γ_{ijk} and η_{ijk} represents the degree of satisfaction and degree of dissatisfaction respectively of the decision maker with respect to the cost c_{ijk} required for transporting one unit quantity of the product from the i^{th} source S_i to the j^{th}

destination D_j by the k^{th} vehicle and satisfies the conditions $0 \leq \gamma_{ijk} \leq 1, 0 \leq \eta_{ijk} \leq 1, \gamma_{ijk} + \eta_{ijk} \leq 1$.

(iii) The real-number a_i represents the availability of the product at the i^{th}source S_i.

(iv) The real-number b_j represents the demand of the product at the j^{th}destination D_j.

(v) The natural number m represents the number of available sources.

(vi) The natural number n represents the number of available destinations.

Step 1: Transform the dual-hesitant fuzzy transportation problem (represented by Table 2.1) into its equivalent crisp transportation problem (represented by Table 2.2).

Table 2.2. Transformed crisp transportation problem

	D_1	D_2	$\cdots$	D_n	a_i
S_1	$Score(\tilde{c}_{11})$	$Score(\tilde{c}_{12})$	$\cdots$	$Score(\tilde{c}_{1n})$	a_1
S_2	$Score(\tilde{c}_{21})$	$Score(\tilde{c}_{22})$	$\cdots$	$Score(\tilde{c}_{2n})$	a_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
S_m	$Score(\tilde{c}_{m1})$	$Score(\tilde{c}_{m2})$	$\cdots$	$Score(\tilde{c}_{mn})$	a_m
b_j	b_1	b_2	$\cdots$	b_n	

where,

$$Score\left(\tilde{c}_{ij}\right) = Score\left(\begin{Bmatrix}\{\gamma_{ij1}, \gamma_{ij2}, \dots, \gamma_{ijk}, \dots, \gamma_{ijp}\},\\ \{\eta_{ij1}, \eta_{ij2}, \dots, \eta_{ijk}, \dots, \eta_{ijp}\}\end{Bmatrix}\left(\{c_{ij1}, c_{ij2}, \dots, c_{ijk}, \dots, c_{ijp}\}\right)\right)$$

$$= \left| \frac{1}{p} \sum_{k=1}^{p} \gamma_{ijk} - \frac{1}{p} \sum_{k=1}^{p} \eta_{ijk} \right|$$

Step 2: Find the optimal solution $\{x_{ij}, i = 1,2, \ldots, m; j = 1,2, \ldots, n\}$ of the transformed crisp transportation problem (represented by Table 2.2) by using classical methods [1, 5, 37]. The obtained optimal solution represents the optimal solution of the dual-hesitant fuzzy transportation problem (represented by Table 2.1).

Step 3: Using the optimal solution $\{x_{ij}, i = 1,2, \ldots, m; j = 1,2, \ldots, n\}$, obtained in Step 2, find the optimal transportation cost $\sum_{i=1}^{m} \sum_{j=1}^{n} \frac{(c_{ij1}+c_{ij2}+\cdots+c_{ijk}+\cdots+c_{ijp})x_{ij}}{p}$.

2.2. **Inappropriateness of Maity et al.'s approach**

It is inappropriate to use Maity et al.'s approach [29] to solve dual-hesitant fuzzy transportation problems due to the following reasons:

1. Maity et al.'s approach [3] fails to find the optimal solution of a dual-hesitant fuzzy transportation problem. To validate this claim, the dual-hesitant fuzzy transportation problem, considered in Example 2.1, is solved by Maity et al.'s approach [29] and shown that the obtained solution is not an optimal solution.

   Example 2.1 A supplier needs to supply the milk from two plants S_1 and S_2 to two places D_1 and D_2. The availability $a_i (i = 1,2)$ of the milk at sources S_i $(i = 1,2)$, the demands b_j $(j = 1,2)$ of the milk at destinations D_j $(j = 1,2)$ and the dual-hesitant fuzzy transportation cost for supplying one unit (100 liter) quantity of milk from sources S_i $(i = 1,2)$ to destinations D_j $(j = 1,2)$ are mentioned in Table 2.3. The supplier is interested to find the optimal way for supplying the milk and the corresponding associated minimum transportation cost.

Table 2.3. Dual-hesitant fuzzy transportation problem

	D_1	D_2	a_i
S_1	$\begin{pmatrix} \begin{Bmatrix} \{0.5, 0.4, 0.3\}, \\ \{0.1, 0.2, 0.3\} \end{Bmatrix} \\ (70,80,90) \end{pmatrix}$	$\begin{pmatrix} \begin{Bmatrix} \{0.8, 0.7, 0.6\}, \\ \{0.1, 0.2, 0.3, \} \end{Bmatrix} \\ (10,20,30) \end{pmatrix}$	20
S_2	$Score\begin{pmatrix} \begin{Bmatrix} \{0.8, 0.6\}, \\ \{0.1, 0.3\} \end{Bmatrix} \\ (5,15) \end{pmatrix}$	$Score\begin{pmatrix} \begin{Bmatrix} \{0.6, 0.5\}, \\ \{0.2, 0.3\} \end{Bmatrix} \\ (80,90) \end{pmatrix}$	20
b_j	20	20	

Using Maity et al.'s approach [29], the optimal way for supplying the milk and the corresponding associated minimum transportation cost for the dual-hesitant fuzzy transportation problem (represented by Table 2.3) can be obtained as follows:

Step 1: Using Step 1 of Maity et al.'s approach [29], the dual-hesitant fuzzy transportation problem (represented by Table 2.3) can be transformed into its equivalent crisp transportation problem (represented by Table 2.4).

Table 2.4. Transformed crisp transportation problem

	D_1	D_2	a_i
S_1	$Score\begin{pmatrix}\begin{Bmatrix}\{0.5, 0.4, 0.3\},\\ \{0.1, 0.2, 0.3\}\end{Bmatrix}\\ (70,80,90)\end{pmatrix} = \left\lvert\frac{0.5+0.4+0.3}{3} - \frac{0.1+0.2+0.3}{3}\right\rvert = 0.2$	$Score\begin{pmatrix}\begin{Bmatrix}\{0.8, 0.7, 0.6\},\\ \{0.1, 0.2, 0.3, \}\end{Bmatrix}\\ (10,20,30)\end{pmatrix} = \left\lvert\frac{0.8+0.7+0.6}{3} - \frac{0.1+0.2+0.3}{3}\right\rvert = 0.5$	20
S_2	$Score\begin{pmatrix}\begin{Bmatrix}\{0.8, 0.6\},\\ \{0.1, 0.3\}\end{Bmatrix}\\ (5,15)\end{pmatrix} = \left\lvert\frac{0.8+0.6}{2} - \frac{0.1+0.3}{2}\right\rvert = 0.5$	$Score\begin{pmatrix}\begin{Bmatrix}\{0.6, 0.5\},\\ \{0.2, 0.3\}\end{Bmatrix}\\ (80,90)\end{pmatrix} = \left\lvert\frac{0.6+0.5}{2} - \frac{0.2+0.3}{2}\right\rvert = 0.3$	20
b_j	20	20	

Step 2: On solving the crisp transportation problem (represented by Table 2.4), the following optimal solution is obtained.

$x_{11} = 20,\ x_{12} = 0, x_{21} = 0, x_{22} = 20.$

Step 3: Using Step 3 of Maity et al.'s approach, the optimal transportation cost is $\frac{(70+80+90)(20)}{3} + \frac{(10+20+30)(0)}{3} + \frac{(5+15)(0)}{2} + \frac{(80+90)(20)}{2} = 3300.$

It is obvious that according to Maity et al.'s approach [29], the optimal solution of the dual-hesitant fuzzy transportation problem (represented by Table 2.3) is $x_{11} = 20,\ x_{12} = 0, x_{21} = 0, x_{22} = 20.$

While, it is not the optimal solution as the total transportation cost of the crisp transportation problem (represented by Table 2.4) corresponding to the feasible solution $x_{11} = 0,\ x_{12} = 20, x_{21} = 20, x_{22} = 0$ i.e., $\frac{(70+80+90)(0)}{3} + \frac{(10+20+30)(20)}{3} + \frac{(5+15)(20)}{2} + \frac{(80+90)(0)}{2} = 600$ is less than 3300.

This clearly indicates that Maity et al.'s approach [29] fails to find the optimal solution of the considered dual-hesitant fuzzy transportation problem.

2. On applying Maity et al.'s approach [29] different optimal transportation cost is obtained corresponding to alternative optimal solutions. This contradicts the well-known fact that the optimal transportation cost corresponding to all the possible alternative optimal solutions should be same. To validate this claim, the dual-hesitant

fuzzy transportation problem, considered in Example 2.2, is solved by Maity et al.'s approach.

Example 2.2 A supplier needs to supply the milk from two plants S_1 and S_2 to two places D_1 and D_2. The availability $a_i (i = 1,2)$ of the milk at sources S_i $(i = 1,2)$, the demands b_j $(j = 1,2)$ of the milk at destinations D_j $(j = 1,2)$ and the dual-hesitant fuzzy transportation cost for supplying one unit (100 liter) quantity of milk from sources S_i $(i = 1,2)$ to destinations D_j $(j = 1,2)$ are mentioned in Table 2.5. The supplier is interested to find the optimal way for supplying the milk and the corresponding associated minimum transportation cost.

Table 2.5. Dual-hesitant fuzzy transportation problem

	D_1	D_2	a_i
S_1	$\left\{\begin{matrix}\{0.5, 0.4, 0.3\},\\ \{0.1, 0.2, 0.3\}\end{matrix}\right\}$ $(70,80,90)$	$\left\{\begin{matrix}\{0.6, 0.5, 0.4\},\\ \{0.2, 0.3, 0.4\}\end{matrix}\right\}$ $(10,20,30)$	20
S_2	$\left\{\begin{matrix}\{0.6, 0.4\},\\ \{0.2, 0.4\}\end{matrix}\right\}$ $(5,15)$	$\left\{\begin{matrix}\{0.6, 0.5\},\\ \{0.3, 0.4\}\end{matrix}\right\}$ $(80,90)$	20
b_j	20	20	

Using Maity et al.'s approach [29], the optimal way for supplying the milk and the corresponding associated minimum transportation cost for the dual-hesitant fuzzy transportation (represented by Table 2.5) can be obtained as follows:

Step 1: Using Step 1 of Maity et al.'s approach, the dual-hesitant fuzzy transportation problem (represented by Table 2.5) can be transformed into its equivalent crisp transportation problem (represented by Table 2.6).

Table 2.6. Transformed crisp transportation problem

	D_1	D_2	a_i
S_1	$Score\begin{pmatrix}\begin{Bmatrix}\{0.5, 0.4, 0.3\}, \\ \{0.1, 0.2, 0.3\}\end{Bmatrix} \\ (70,80,90)\end{pmatrix}$ $= \left\lvert\frac{0.5 + 0.4 + 0.3}{3} - \frac{0.1 + 0.2 + 0.3}{3}\right\rvert = 0.2$	$Score\begin{pmatrix}\begin{Bmatrix}\{0.6, 0.5, 0.4\}, \\ \{0.2, 0.3, 0.4\}\end{Bmatrix} \\ (10,20,30)\end{pmatrix}$ $= \left\lvert\frac{0.6 + 0.5 + 0.4}{3} - \frac{0.2 + 0.3 + 0.4}{3}\right\rvert = 0.2$	20

S_2	$Score\begin{pmatrix}\begin{Bmatrix}\{0.6, 0.4\},\\ \{0.2, 0.4\}\end{Bmatrix}\\ (5,15)\end{pmatrix}$ $= \left\|\frac{0.6+\ 0.4}{2} - \frac{0.2+\ 0.4}{2}\right\|$ $= 0.2$	$Score\begin{pmatrix}\begin{Bmatrix}\{0.6, 0.5\},\\ \{0.3, 0.4\}\end{Bmatrix}\\ (80,90)\end{pmatrix}$ $\left\|\frac{0.6+\ 0.5}{2} - \frac{0.3+\ 0.4}{2}\right\|$ $= 0.2$	20
b_j	20	20	

Step 2: On solving the crisp transportation problem (represented by Table 2.6), the following two optimal basic feasible solutions are obtained:

(i) $x_{11} = 20,\ x_{12} = 0, x_{21} = 0, x_{22} = 20.$

(ii) $x_{11} = 0,\ x_{12} = 20, x_{21} = 20, x_{22} = 0.$

Step 3: Using Step 3 of Maity et al.'s approach [29], the optimal transportation cost corresponding to

(i) The first optimal basic feasible solution $x_{11} = 20,\ x_{12} = 0, x_{21} = 0, x_{22} = 20$is $\frac{(70+80+90)(20)}{3} + \frac{(10+20+30)(0)}{3} + \frac{(5+15)(0)}{2} + \frac{(80+90)(20)}{2} = 3300.$

(ii) The second optimal basic feasible solution $x_{11} = 0,\ x_{12} = 20, x_{21} = 20, x_{22} = 0$ is $\frac{(70+80+90)(0)}{3} + \frac{(10+20+30)(20)}{3} + \frac{(5+15)(20)}{2} + \frac{(80+90)(20)}{2} = 600.$

It is obvious that, on applying Maity et al.'s approach [29], different optimal transportation cost is obtained corresponding to alternative optimal solutions, which is mathematically incorrect.

3. Maity et al. [29] claimed that as the optimal transportation cost of a dual-hesitant fuzzy transportation problem with score value i.e., by their proposed approach will lie between the optimal transportation cost of dual-hesitant fuzzy transportation problem with minimum hesitant fuzzy cost and the optimal transportation cost of a dual-hesitant fuzzy transportation problem with maximum hesitant fuzzy cost. Therefore, the optimal solution, obtained by their proposed approach, is the best optimal solution. However, in actual case, this condition will not necessarily be satisfied. To validate this claim, the optimal transportation cost of the dual-hesitant fuzzy transportation problem, considered in Example 2.1, is obtained by considering the minimum hesitant fuzzy cost and maximum hesitant fuzzy cost.

   The optimal transportation cost of the dual-hesitant fuzzy transportation problem, considered in Example 2.1, by considering the minimum hesitant fuzzy cost and maximum hesitant fuzzy cost can be obtained as follows:

   Step 1: The crisp transportation problem (represented by Table 2.7) represents the dual-hesitant fuzzy transportation problem with minimum hesitant fuzzy cost and the crisp transportation problem (represented by Table 2.8) represents the dual-hesitant fuzzy transportation problem with maximum hesitant fuzzy cost corresponding to the dual-hesitant fuzzy transportation problem (represented by Table 2.3).

Table 2. 7 Transportation problem with minimum hesitant fuzzy cost

	D_1	D_2	a_i
S_1	70	10	20
S_2	5	80	20
b_j	20	20	

Table 2.8 Transportation problem with maximum hesitant fuzzy cost

	D_1	D_2	a_i
S_1	90	30	20
S_2	15	90	20
b_j	20	20	

Step 2: On solving the crisp transportation problem with minimum hesitant fuzzy cost (represented by Table 2.7), the obtained optimal solution is $x_{11} = 0,\ x_{12} = 20, x_{21} = 20, x_{22} = 0$ and the corresponding optimal transportation cost is 300. Also, on solving the crisp transportation problem with maximum hesitant fuzzy cost (represented by Table 2.8), the obtained optimal solution is $x_{11} = 0,\ x_{12} = 20, x_{21} = 20, x_{22} = 0$ and the corresponding optimal transportation cost is 900.

Furthermore, it is obvious from Step 3 of Example 2.1 that on solving the crisp transportation problem with score value (represented by Table 2.4), the obtained optimal transportation cost is 3300.

It is obvious that the optimal transportation cost of the transportation problem with score value i.e., 3300 does not lie between the optimal transportation cost of the dual-hesitant fuzzy transportation problem with minimum hesitant fuzzy cost i.e., 300 and the optimal transportation cost of the dual-hesitant fuzzy transportation problem with maximum hesitant fuzzy cost i.e., 900.

2.3. Proposed Mehar score function

It is obvious that the expression

$$Score\left(\tilde{c}_{ij}\right) = Score\left(\begin{pmatrix}\{\gamma_{ij1}, \gamma_{ij2}, \dots, \gamma_{ijk}, \dots, \gamma_{ijp}\}, \\ \{\eta_{ij1}, \eta_{ij2}, \dots, \eta_{ijk}, \dots, \eta_{ijp}\}\end{pmatrix}\left(\{c_{ij1}, c_{ij2}, \dots, c_{ijk}, \dots, c_{ijp}\}\right)\right)$$

$= \left|\frac{1}{p}\sum_{k=1}^{p}\gamma_{ijk} - \frac{1}{p}\sum_{k=1}^{p}\eta_{ijk}\right|$, used by Maity et al. [29], is independent from the values of $c_{ij1}, c_{ij2}, \dots, c_{ijk}, \dots, c_{ijp}$. Due to the same reason, Maityet al.'s approach [29] fails to find the appropriate solution of the considered dual-hesitant fuzzy transportation problems. This expression is proposed by considering the assumption that the decision maker would like to maximize the value of $\frac{1}{p}\sum_{k=1}^{p}\gamma_{ijk}$ and to minimize the value of $\frac{1}{p}\sum_{k=1}^{p}\eta_{ijk}$ simultaneously. While, in actual case, the decision maker would like to minimize the value of $\frac{1}{p}\sum_{k=1}^{p}c_{ijk}$, to maximize the value of $\frac{1}{p}\sum_{k=1}^{p}\gamma_{ijk}$ and to minimize the value of $\frac{1}{p}\sum_{k=1}^{p}\eta_{ijk}$ simultaneously. Therefore, it is appropriate to use the following expression (named as Mehar score function) to transform a dual-hesitant fuzzy set into a real number instead of using the existing expression:

$$MScore\left(\tilde{c}_{ij}\right) = MScore\left(\begin{Bmatrix}\{\gamma_{ij1},\gamma_{ij2},\dots,\gamma_{ijk},\dots,\gamma_{ijp}\},\\ \{\eta_{ij1},\eta_{ij2},\dots,\eta_{ijl},\dots,\eta_{ijp}\}\end{Bmatrix}(\{c_{ij1},c_{ij2},\dots,c_{ijk},\dots,c_{ijp}\})\right)$$

$$= \frac{1}{p}\sum_{k=1}^{p} c_{ijk} + \frac{1}{p}\sum_{k=1}^{p} \eta_{ijk} - \frac{1}{p}\sum_{k=1}^{p} \gamma_{ijk}$$

2.4. Proposed Mehar approach

It is obvious from Section 2.2 that it is not appropriate to use Maity et al.'s approach [29] is to find the optimal solution of dual-hesitant fuzzy transportation problems.

In this section, an appropriate approach (named as Mehar approach) is proposed to find the optimal solution of the dual-hesitant fuzzy transportation problems.

The steps of the proposed Mehar approach are as follows:

Step 1: Transform the dual-hesitant fuzzy transportation problem (represented by Table 2.1) into the crisp transportation problem represented by Table 2.9.

Table 2.9. Transformed crisp transportation problem

	D_1	D_2	$\cdots$	D_n	Availability
S_1	$MScore(\tilde{c}_{11})$	$MScore(\tilde{c}_{12})$	$\cdots$	$MScore(\tilde{c}_{1n})$	a_1
S_2	$MScore(\tilde{c}_{21})$	$MScore(\tilde{c}_{22})$	$\cdots$	$MScore(\tilde{c}_{2n})$	a_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
S_m	$MScore(\tilde{c}_{m1})$	$MScore(\tilde{c}_{m2})$	$\cdots$	$MScore(\tilde{c}_{mn})$	a_m
Demand	b_1	b_2	$\cdots$	b_n	

where,

$$MScore\left(\tilde{c}_{ij}\right) = \frac{1}{p}\sum_{k=1}^{p} c_{ijk} + \frac{1}{p}\sum_{k=1}^{p} \eta_{ijk} - \frac{1}{p}\sum_{k=1}^{p} \gamma_{ijk}$$

Step 2: Find the optimal solution $\{x_{ij}, i = 1,2, \dots, m; j = 1,2, \dots, n\}$ and the optimal transportation of the transformed crisp transportation problem (represented by Table 2.9). The obtained optimal solution and the obtained optimal transportation cost represents the optimal solution and optimal transportation cost respectively of the dual-hesitant fuzzy transportation problem (represented by Table 2.1).

Remark 1: In the proposed Mehar approach, the considered dual-hesitant fuzzy transportation problem is firstly transformed into its equivalent crisp transportation problem. Then, the optimal solution of the transformed crisp transportation problem is obtained. Since, there exist several methods in the literature to solve a crisp transportation problem and hence, different researchers may use different methods to solve the transformed crisp transportation problem. Therefore, the computational time complexity and accuracy performance of the proposed Mehar approach will be same as the computational time complexity and accuracy performance of that approach which will be used to solve the transformed crisp transportation problem.

2.5. Optimal solutions of the considered dual-hesitant fuzzy transportation problems

In Section 2.2, two dual-hesitant fuzzy transportation problems (represented by Table 2.3 and Table 2.5) are solved by Maity et al.'s approach [29] and pointed out Maity et al.'s approach fails to find the optimal solutions of the considered dual-hesitant fuzzy transportation problems. In this section, the appropriate solutions of these problems are obtained by the proposed Mehar approach.

2.5.1 Optimal solution of the first dual-hesitant fuzzy transportation problem

Using the proposed Mehar approach, the optimal solution of the first dual-hesitant fuzzy transportation problem (represented by Table 2.3) can be obtained as follows:

Step 1: Using Step 1 of the Mehar approach, proposed in Section 2.4, the dual-hesitant fuzzy transportation problem (represented by Table 2.3) can be transformed into its equivalent crisp transportation problem (represented by Table 2.10).

	D_1	D_2	a_i
S_1	$MScore\begin{pmatrix}\left\{\begin{matrix}\{0.5,0.4,0.3\},\\ \{0.1,0.2,0.3\}\end{matrix}\right\}\\ (70,80,90)\end{pmatrix}$ $=\frac{70+80+90}{3}+\frac{0.1+0.2+0.3}{3}-\frac{0.5+0.4+0.3}{3}$ $=79.8$	$MScore\begin{pmatrix}\left\{\begin{matrix}\{0.8,0.7,0.6\},\\ \{0.1,0.2,0.3,\}\end{matrix}\right\}\\ (10,20,30)\end{pmatrix}$ $=\frac{10+20+30}{3}+\frac{0.1+0.2+0.3}{3}-\frac{0.8+0.7+0.6}{3}$ $=19.5$	20
S_2	$MScore\begin{pmatrix}\left\{\begin{matrix}\{0.8,0.6\},\\ \{0.1,0.3\}\end{matrix}\right\}\\ (5,15)\end{pmatrix}$ $=\frac{5+15}{2}+\frac{0.1+0.3}{2}-\frac{0.8+0.6}{2}$ $=9.5$	$MScore\begin{pmatrix}\left\{\begin{matrix}\{0.6,0.5\},\\ \{0.2,0.3\}\end{matrix}\right\}\\ (80,90)\end{pmatrix}$ $=\frac{80+90}{2}+\frac{0.2+0.3}{2}-\frac{0.6+0.5}{2}$ $=84.7$	20
b_j	20	20	

Table 2.10 Transformed crisp transportation problem

Step 2: On solving the crisp transportation problem (represented by Table 2.10), the obtained optimal solution is $x_{11} = 0$, $x_{12} = 20, x_{21} = 20, x_{22} = 0$ and the obtained optimal transportation cost is 580.

2.5.2 Optimal solution of the second dual-hesitant fuzzy transportation problem

Using the proposed Mehar approach, the optimal solution of the second dual-hesitant fuzzy transportation problem (represented by Table 2.5) can be obtained as follows:

Step 1: Using Step 1 of the Mehar approach, proposed in Section 2.3, the dual-hesitant fuzzy transportation problem (represented by Table 2.5) can be transformed into its equivalent crisp transportation problem (represented by Table 2.11)

Table 2.11 Transformed crisp transportation problem

	D_1	D_2	a_i
S_1	$MScore\left(\begin{matrix}\left\{\begin{matrix}\{0.5, 0.4, 0.3\},\\ \{0.1, 0.2, 0.3\}\end{matrix}\right\}\\ (70,80,90)\end{matrix}\right)$ $= \frac{70+80+90}{3}$ $+ \frac{0.1 + 0.2 + 0.3}{3}$ $- \frac{0.5 + 0.4 + 0.3}{3}$ $= 79.8$	$MScore\left(\begin{matrix}\left\{\begin{matrix}\{0.6, 0.5, 0.4\},\\ \{0.2, 0.3, 0.4, \}\end{matrix}\right\}\\ (10,20,30)\end{matrix}\right)$ $= \frac{10+20+30}{3} +$ $\frac{0.2 + 0.3 + 0.4}{3}$ $- \frac{0.6 + 0.5 + 0.4}{3}$ $= 19.8$	20

S_2	$MScore\begin{pmatrix}\begin{Bmatrix}\{0.6,0.4\},\\ \{0.2,0.4\}\end{Bmatrix}\\ (5,15)\end{pmatrix} = \frac{5+15}{2} + \frac{0.2+\ 0.4}{2} - \frac{0.6+\ 0.4}{2} = 9.8$	$MScore\begin{pmatrix}\begin{Bmatrix}\{0.6,0.5\},\\ \{0.3,0.4\}\end{Bmatrix}\\ (80,90)\end{pmatrix} = \frac{80+90}{2} + \frac{0.3+\ 0.4}{2} - \frac{0.6+\ 0.5}{2} = 84.8$	20
b_j	20	20	

Step 2: On solving the crisp transportation problem (represented by Table 2.11), the obtained optimal solution is $x_{11} = 0,\ x_{12} = 20, x_{21} = 20, x_{22} = 0$ and the obtained optimal transportation cost is 592.

2.6 Conclusion

It is pointed out that Maity et al.'s approach [29] is not appropriate. Also, it is pointed out that the inappropriateness of score function, used by Maity et al. [29], is the reason for the inappropriateness of Maity et al.'s approach [29]. Furthermore, to resolve the inappropriateness of Maity et al.'s approach [29], a new expression (named as Mehar score function) and an appropriate approach (named as Mehar approach), based upon the proposed Mehar score function, is proposed. In future, the proposed Mehar approach may be extended to solve generalized dual-hesitant intuitionistic fuzzy multi-objective transportation problems which are the generalization of the existing intuitionistic fuzzy multi-objective transportation problems [31].

CHAPTER 3

A new method for solving Pythagorean fuzzy transportation problems

Kumar et al. [22] proposed a new procedure for solving the Pythagorean fuzzy transportation problems. In their method, to defuzzify Pythagorean fuzzy cost of each cell, score function was calculated by using only the degree of membership and degree of non-membership values. This score function is independent from the value of cost of each cell which has no meaning. So keeping it in mind, a new score function named as PS score function, is proposed to find the optimal solution of Pythagorean fuzzy transportation problem.

3.1 Kumar et al.'s method for solving Pythagorean fuzzy transportation problems

Kumar et al. [22] proposed the following method to find the optimal solution of Pythagorean fuzzy transportation problem (represented by Table 3.1).

Table 3.1: Pythagorean fuzzy transportation problem

Destinations	D_1	D_2	$\cdots$	D_n	Availability

Sources					
S_1	$\widetilde{c_{11}^P}$	$\widetilde{c_{12}^P}$	$\cdots$	$\widetilde{c_{1n}^P}$	a_1
S_2	$\widetilde{c_{21}^P}$	$\widetilde{c_{22}^P}$	$\cdots$	$\widetilde{c_{2n}^P}$	a_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
S_m	$\widetilde{c_{m1}^P}$	$\widetilde{c_{m2}^P}$	$\cdots$	$\widetilde{c_{mn}^P}$	a_m
Demand	b_1	b_2	$\cdots$	b_n	

where,

The Pythagorean fuzzy set $\widetilde{c_{ij}^P} = \left(\theta_{ij}^P\ ,\ \delta_{ij}^P\right)$

(i) represents the cost for supplying the unit quantity of the product from the i^{th} source S_i to the j^{th} destination D_j.

(ii) θ_{ij}^P and δ_{ij}^P represents the degree of membership and degree of non-membership respectively of the decision maker with respect to the cost c_{ij} required for transporting one unit quantity of the product from the i^{th} source S_i to the j^{th} destination D_j and satisfies the conditions $0 \leq \theta_{ij}^P \leq 1,\ 0 \leq \delta_{ij}^P \leq 1,\ \left(\theta_{ij}^P\right)^2 + \left(\delta_{ij}^P\right)^2 \leq 1$.

(iii) The real-number a_i represents the availability of the product at the i^{th}source S_i.

(iv) The real-number b_j represents the demand of the product at the j^{th}destination D_j.

(v) The natural number m represents the number of available sources.

(vi) The natural number n represents the number of available destinations.

Step 1: Transform the Pythagorean fuzzy transportation problem (represented by Table 3.1) into its equivalent crisp transportation problem (represented by Table 3.2).

Step 2: Find the optimal solution $\{x_{ij}, i = 1,2, \ldots, m; j = 1,2, \ldots, n\}$ of the transformed crisp transportation problem (represented by Table 3.2). The obtained optimal solution represents the optimal solution of the Pythagorean fuzzy transportation problem (represented by Table 3.1).

Table 3.2: Transformed crisp transportation problem

Destinations / Sources	D_1	D_2	$\cdots$	D_n	Availability
S_1	$S\left(\widetilde{c_{11}^P}\right)$	$S\left(\widetilde{c_{12}^P}\right)$	$\cdots$	$S\left(\widetilde{c_{1n}^P}\right)$	a_1
S_2	$S\left(\widetilde{c_{21}^P}\right)$	$S\left(\widetilde{c_{22}^P}\right)$	$\cdots$	$S\left(\widetilde{c_{2n}^P}\right)$	a_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
S_m	$S\left(\widetilde{c_{m1}^P}\right)$	$S\left(\widetilde{c_{m2}^P}\right)$	$\cdots$	$S\left(\widetilde{c_{mn}^P}\right)$	a_m
Demand	b_1	b_2	$\cdots$	b_n	

where, $S\left(\widetilde{c_{ij}^P}\right) = \frac{1}{2}\left[1 + \left(\left(\theta_{ij}^P\right)^2 - \left(\delta_{ij}^P\right)^2\right)\right]$

Step 3: Using the optimal solution $\{x_{ij}, i = 1,2, \ldots, m; j = 1,2, \ldots, n\}$, obtained in Step 2, find the optimal transportation cost $\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{(c_{ij1}+c_{ij2}+\cdots+c_{ijk}+\cdots+c_{ijp})x_{ij}}{p}$.

3.2. Inappropriateness of Kumar et al.'s score function

It is obvious that the expression $\left(\widetilde{c_{ij}^P}\right) = S\left(\theta_{ij}^P\ ,\ \delta_{ij}^P\right) = \frac{1}{2}\left[1 + \left(\left(\theta_{ij}^P\right)^2 - \left(\delta_{ij}^P\right)^2\right)\right]$, used by Kumar et al. [22], is independent from the values of c_{ij}. Due to same reason, Kumar et al.'s method [22] fails to find the appropriate solution of considered Pythagorean fuzzy transportation problems.

To validate this claim, the existing Pythagorean fuzzy transportation problem (Example 5.1 [22]), represented by Table 3.3, is solved by Kumar et al.'s [22] method.

Table 3.3: Pythagorean fuzzy transportation problem

Sources \ Destinations	D_1	D_2	D_3	D_4	Supply
S_1	30 $(0.4, 0.7)$	20 $(0.5, 0.4)$	25 $(0.8, 0.3)$	20 $(0.6, 0.3)$	26
S_2	15 $(0.4, 0.2)$	35 $(0.7,0.3)$	22 $(0.4,0.8)$	25 $(0.7, 0.3)$	24
S_3	35 $(0.7,0.1)$	25 $(0.8, 0.1)$	26 $(0.6, 0.4)$	40 $(0.9, 0.1)$	30
Demand	17	23	28	12	

Using Kumar et al.'s method [22], the optimal minimum transportation cost for the Pythagorean fuzzy transportation problem (represented by Table 3.3) can be obtained as follows:

Step 1: Using Step 1 of Kumar et al.'s method [22], the Pythagorean fuzzy transportation problem (represented by Table 3.3) can be transformed into its equivalent crisp transportation problem (represented by Table 3.4).

Table 3.4: Transformed crisp transportation problem

Sources \ Destinations	D_1	D_2	D_3	D_4	Supply
S_1	$S\,(0.4, 0.7) = 0.335$	$S\,(0.5, 0.4) = 0.545$	S $(0.8, 0.3) = 0.775$	$S\,(0.6, 0.3) = 0.635$	26
S_2	$S\,(0.4, 0.2) = 0.56$	$S(0.7,0.3) = 0.7$	$S\,(0.4,0.8) = 0.26$	$S\,(0.7, 0.3) = 0.7$	24
S_3	$S(0.7,0.1) = 0.74$	$S(0.8, 0.1) = 0.815$	$S(0.6, 0.4) = 0.6$	$S(0.9, 0.1) = 0.9$	30
Demand	17	23	28	12	

Step 2: On solving the transformed transportation problem, represented by Table 3.4, the obtained optimal solution is $x_{11} = 17,\ x_{12} = 9, x_{23} = 24,\ x_{32} = 14,\ x_{33} = 4,\ x_{34} = 12.$

Step 3: Using Step 3 of Kumar et al.'s method [22], the total minimum transportation cost is $17 \times 0.335 + 9 \times 0.545 + 24 \times 0.26 + 14 \times 0.815 + 4 \times 0.6 + 12 \times 0.9 = 41.45$

3.3. Proposed Score function

It is obvious that the expression $S\left(\widetilde{c_{ij}^{P}}\right) = S\left(\theta_{ij}^{P}\ ,\ \delta_{ij}^{P}\right) = \frac{1}{2}\left[1 + \left(\left(\theta_{ij}^{P}\right)^2 - \left(\delta_{ij}^{P}\right)^2\right)\right]$, used by Kumar et al. [22]. is independent from the values of c_{ij} .

Therefore the following expression (name as PS score function) is proposed to transform Pythagorean fuzzy set into a real number instead of using the existing expression

$$PS\left(\widetilde{c_{ij}^{P}}\right) = PS\left(\theta_{ij}^{P}, \delta_{ij}^{P}\right) = c_{ij} + \frac{1}{2}\left[1 + \left(\left(\theta_{ij}^{P}\right)^2 - \left(\delta_{ij}^{P}\right)^2\right)\right]$$

3.4. Proposed Method

It is obvious from Section 3.2 that it is not appropriate to use Kumar et. al.'s method [22] to find the optimal solution of Pythagorean fuzzy transportation problems.

In this section, an appropriate method is proposed to find the optimal solution of Pythagorean fuzzy transportation problems by using proposed score function.

The steps of the proposed method are as follows:

Step 1: Transform the Pythagorean fuzzy transportation problem (represented by Table 3.1) into the crisp transportation problem (represented by Table 3.5) using proposed score function.

Table 3.5: Transformed crisp transportation problem

Destinations / Sources	D_1	D_2	$\cdots$	D_n	Availability
S_1	$PS\left(\widetilde{c_{11}^P}\right)$	$PS\left(\widetilde{c_{12}^P}\right)$	$\cdots$	$PS\left(\widetilde{c_{1n}^P}\right)$	a_1
S_2	$PS\left(\widetilde{c_{21}^P}\right)$	$PS\left(\widetilde{c_{22}^P}\right)$	$\cdots$	$PS\left(\widetilde{c_{2n}^P}\right)$	a_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
S_m	$PS\left(\widetilde{c_{m1}^P}\right)$	$PS\left(\widetilde{c_{m2}^P}\right)$	$\cdots$	$PS\left(\widetilde{c_{mn}^P}\right)$	a_m
Demand	b_1	b_2	$\cdots$	b_n	

where,

$$PS\left(\widetilde{c_{ij}^{P}}\right) = PS\left(\theta_{ij}^{P}\ ,\ \delta_{ij}^{P}\right) = c_{ij} + \frac{1}{2}\left[1 + \left(\left(\theta_{ij}^{P}\right)^{2} - \left(\delta_{ij}^{P}\right)^{2}\right)\right]$$

Step 2: Find the optimal solution $\{x_{ij}, i = 1,2,\ldots,m;\ j = 1,2,\ldots,n\}$ of the transformed crisp transportation problem (represented by Table 3.5). The obtained optimal solution represents the optimal solution of the Pythagorean fuzzy transportation problem (represented by Table 3.1) and the obtained transportation cost represents the optimal transportation cost.

3.5. Optimal solution of the Pythagorean fuzzy transportation problem

Using the proposed method, the optimal solution of the Pythagorean fuzzy transportation problem, represented by Table 3.3, can be obtained as follows:

Step 1: Using Step 1 of the proposed method, proposed in Section 3.4, the Pythagorean fuzzy transportation problem (represented by Table 3.3) can be transformed into its equivalent crisp transportation problem (represented by Table 3.6).

Table 3.6: Transformed crisp transportation problem

Sources \ Destinations	D_1	D_2	D_3	D_4	Supply
S_1	PS(30 $(0.4, 0.7)$)= $30 + \frac{1}{2}[1 + ((0.4)^2 - (0.7)^2)]$ = 30.335	PS(20 $(0.5, 0.4)$)= $20 + \frac{1}{2}[1 + ((0.5)^2 - (0.4)^2)]$= 20.545	PS(25 $(0.8, 0.3)$)= $25 + \frac{1}{2}[1 + ((0.8)^2 - (0.3)^2)]$= 25.775	PS(20 $(0.6, 0.3)$)= $20 + \frac{1}{2}[1 + ((0.6)^2 - (0.3)^2)]$= 20.635	26
S_2	PS(15 $(0.4, 0.2)$)= $15 + \frac{1}{2}[1 + ((0.4)^2 - (0.2)^2)]$=15.56	PS(35 (0.7,0.3))= $35 + \frac{1}{2}[1 + ((0.7)^2 - (0.3)^2)]$=35.7	PS(22 (0.4,0.8)) = $22 + \frac{1}{2}[1 + ((0.4)^2 - (0.8)^2)]$=22.26	PS(25 $(0.7, 0.3)$) = $25 + \frac{1}{2}[1 + ((0.7)^2 - (0.3)^2)]$= 25.7	24
S_3	PS(35 (0.7,0.1))= $35 + \frac{1}{2}[1 +$	PS(25 $(0.8, 0.1)$) =	PS(26 $(0.6, 0.4)$)= $26 + \frac{1}{2}[1 +$	PS(40 $(0.9, 0.1)$) = $40 + \frac{1}{2}[1 +$	30

	$((0.7)^2 - (0.1)^2)]=$ 35.74	$25 + \frac{1}{2}[1 + ((0.8)^2 - (0.1)^2)]=$ 25.815	$((0.6)^2 - (0.4)^2)]=26.6$	$((0.9)^2 - (0.1)^2)] =$ 40.9	
Demand	17	23	28	12	

Step 2: On solving the crisp transportation problem (represented by Table 3.6), the obtained optimal solution is $x_{12} = 14,\ x_{14} = 12,\ x_{21} = 17, x_{23} = 7, x_{32} = 9, x_{33} = 21$ and the obtained optimal transportation cost is 1746.525.

3.6. Conclusion

It is pointed out that Kumar et al.'s method [22] is not appropriate. Also, it is pointed out that the inappropriateness of score function, used by Kumar et al. [22], is the reason for the inappropriateness of Kumar et al.'s method [22]. Furthermore, to resolve the inappropriateness of Kumar et al.'s method [22], a new expression (named as PS score function) and an appropriate method based upon the proposed PS score function, are proposed.

References

1. Ahuja RK, Magnanti TL and Orlin JB, Network Flows, Algorithms and Applications, Pearson Education Limited, 2014.

2. Akilbasha A, Pandian P and Natarajan G, An innovative exact method for solving fully interval integer transportation problems, *Informatics in Medicine Unlocked*, Vol. 11, pp.95–99, 2018.
3. Arora J, An algorithm for interval-valued fuzzy fractional transportation problem, *Skit Research Journal*, Vol. 8, pp.71–75, 2018.
4. Bajalinov EB, Linear-Fractional Programming: Theory, Methods, Applications and Software, Kluwer Academic Publishers, 2003.
5. Bazaraa MS, Jarvis JJ and Sherali HD, Linear Programming and Network Flows, Wiley, New York, 2014.
6. Bharati SK and Singh SR, Transportation problem under interval-valued intuitionistic fuzzy environment, *International Journal of Fuzzy System*, Vol. 20, pp.1511–1522, 2018.
7. Bustince H, Barrenechea E, Pagola M, Fernandez J, Xu Z, Bedregal B, Montero J, Hagras H, Herrera F and Baets BD, A historical account of types of fuzzy sets and their relationships, *IEEE Transactions on Fuzzy Systems*, Vol. 24, pp.179-194, 2016.
8. Chanas S, Kołodziejczyk W and Machaj A, A fuzzy approach to the transportation problem, *Fuzzy Sets and Systems*, Vol.13, pp. 211–221, 1984.
9. Ebrahimnejad A, A simplified new approach for solving fuzzy transportation problems with generalized trapezoidal fuzzy numbers, *Applied Soft Computin*g, Vol. 19, pp.171-176, 2014.
10. Ebrahimnejad A, New method for solving fuzzy transportation problems with LR flat fuzzy numbers, *Information Sciences*, Vol. 357, pp.108-124, 2016.

11. Ebrahimnejad A, Fuzzy linear programming approach for solving transportation problems with interval-valued trapezoidal fuzzy numbers. *Sadhana*, Vol. 41, pp.299-316, 2016.

12. Ebrahimnejad A and Verdegay JL, A new approach for solving fully intuitionistic fuzzy transportation problems, Fuzzy Optimization and decision making, Vol. 17, pp.447-474, 2018.

13. Gani AN, Samuel AE and Anuradha D, Simplex type algorithm for solving fuzzy transportation problem, *Tamsui Oxford Journal of Information and Mathematical Sciences,* Vol. 27, pp.89–98, 2011.

14. Gupta A and Kumar A, A new method for solving linear multi-objective transportation problems with fuzzy parameters, *Applied Mathematical Modelling*, Vol. 36, pp.1421-1430, 2012.

15. Gupta G and Kumari A, An efficient method for solving intuitionistic fuzzy transportation problem of type-2, *International Journal of Applied and Computational Mathematics*, Vol. 3, pp. 3795–3804, 2017.

16. Hadley G, Linear Programming, Narosa Publishers, 2002.

17. Hitchcock FL, The distribution of a product from several sources to numerous localities, *Journal of Mathematics and Physics*, Vol. 20, pp.224-230, 1941.

18. Kaur A and Kumar A, A new method for solving fuzzy transportation problems using ranking function, *Applied Mathematical Modelling*, Vol. 35, pp.5652–5661, 2011.

19. Kaur A and Kumar A, A new approach for solving fuzzy transportation problem using generalized trapezoidal fuzzy number, *Applied Soft Computing*, Vol. 12, pp.1201-1213, 2012.

20. Klir GJ and Yuan B, Fuzzy Sets & Fuzzy Logic: Theory and Applications, Pearson Education Limited, 2019.

21. Kour D, Mukherjee S and Basu K, Solving intuitionistic fuzzy transportation problem using linear programming, *International Journal of System Assurance Engineering and Management*, Vol. 8, pp. 1090–1101, 2017.

22. Kumar R, Edalatpanah SA, Jha S and Singh R, Pythagorean fuzzy approach to the transportation problem, *Complex & Intelligent Systems*, Vol. 5, pp.255-263, 2019.

23. Kumar PS and Hussain RJ, Computationally simple approach for solving fully intuitionistic fuzzy real life transportation problems simple approach for solving fully intuitionistic fuzzy real life transportation problems, *International Journal of System Assurance Engineering and Management*, Vol. 7, pp.90-101, 2016.

24. Kumar A and Kaur A, Methods for solving unbalanced fuzzy transportation problems, *Operational Research*, Vol. 12, pp.287-316, 2012.

25. Kundu P, Kar S and Maiti M, Fixed charge transportation problem with type-2 fuzzy variables, *Information Sciences*, Vol. 255, pp. 170–186, 2014.

26. Li L and Lai KK, A fuzzy approach to the multi objective transportation problem, *Computers & Operations Research*, Vol. 27, pp. 43–57, 2000.

27. Liu ST and Kao C, Solving fuzzy transportation problems based on extension principle, *European Journal Operational Research*, Vol. 153, pp.661–674, 2004.

28. Liu P, Yang L, Wang L and Li S, A solid transportation problem with type-2 fuzzy variables, *Applied Soft Computing*, Vol. 24, pp. 543–558, 2014.

29. Maity G, Mardanya D, Roy SK and Weber GW, A new approach for solving dual-hesitant fuzzy transportation problem with restrictions, *Sadhana-Academy Proceedings in Engineering Sciences*, https://doi.org/10.1007/s12046-018-1045-1, Vol. 44, 2019.

30. Rani D, Gulati TR and Kumar A, A method for unbalanced transportation problems in fuzzy environment, *Sadhana-Academy Proceedings in Engineering Sciences*, Vol. 39, pp.573-581, 2014.

31. Roy SK, Ebrahimnejad A, Verdegay JL and Das S, New approach for solving intuitionistic fuzzy multi-objective transportation problem, *Sadhana- Academy Proceedings in Engineering Sciences*, https://doi.org/10.1007/s12046-017-0777-7, 2019.

32. Samuel AE and Venkatachalapathy M, Modified Vogel's approximation method for fuzzy transportation problems, *Applied Mathematical Sciences*, Vol. 5, pp.1367–1372, 2011.

33. Singh SK and Yadav SP, Efficient approach for solving type-1 intuitionistic fuzzy transportation problem, *International Journal of System Assurance Engineering and Management*, Vol. 6, pp.259-267, 2014.

34. Singh SK and Yadav SP, A new approach for solving intuitionistic fuzzy transportation problem of type-2, *Annals of Operations Research*, Vol. 243, pp.349-363, 2016.

35. Singh SK and Yadav SP, A novel approach for solving fully intuitionistic fuzzy transportation problem, *International Journal of Operational Research*, 26, pp.460-472, 2016.

36. Tada M and Ishii H, An integer fuzzy transportation problem, *Computers and Mathematics with Applications*, Vol. 31, pp.71–87, 1996.

37. Taha HA, Operations Research: An Introduction, Prentice-Hall, New Jersey, 2003.

38. Zadeh LA, Fuzzy sets, *Information and Control*, Vol. 8, pp.338-353, 1965.